

Magnetic flux tube models in superstring theory

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Abstract

Superstring models describing curved 4-dimensional magnetic flux tube backgrounds are exactly solvable in terms of free fields. We first consider the simplest model of this type (corresponding to ‘Kaluza-Klein’ $a = \sqrt{3}$ Melvin background). Its 2d action has a flat but topologically non-trivial 10-dimensional target space (there is a mixing of angular coordinate of the 2-plane with an internal compact coordinate). We demonstrate that this theory has broken supersymmetry but is perturbatively stable if the radius R of the internal coordinate is larger than $R_0 = \sqrt{2\alpha'}$. In the Green-Schwarz formulation the supersymmetry breaking is a consequence of the presence of a flat but non-trivial connection in the fermionic terms in the action. For $R < R_0$ and the magnetic field strength parameter $q > R/2\alpha'$, there appear instabilities corresponding to tachyonic winding states. The torus partition function $Z(q, R)$ is finite for $R > R_0$ and vanishes for $qR = 2n$ ($n = \text{integer}$). At the special points $qR = 2n(2n+1)$ the model is equivalent to the free superstring theory compactified on a circle with periodic (antiperiodic) boundary condition for space-time fermions. Analogous results are obtained for a more general class of static magnetic flux tube geometries including the $a = 1$ Melvin model.

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1. Introduction

Magnetic backgrounds were actively studied recently from various points of view in the context of both field theory and string theory (see, e.g., [1,2,3,4,5] and [6,7,8,9,10,11,12]). Of particular interest are the simplest ones – static flux tube type configurations with approximately uniform magnetic field generalizing the Melvin solution. Such backgrounds are exact solutions of string theory [10,11] and, moreover, the spectrum of the corresponding conformal string models can be explicitly determined [11]. In the bosonic case these theories are generically unstable due to the appearance of tachyons for certain values of the magnetic field parameters.¹

The problem addressed in the present paper is the construction and solution of the corresponding superstring versions. We shall find that there still exists a range of parameters for which magnetic flux tube backgrounds considered as solutions of superstring theory are perturbatively unstable.

In Section 2 we shall review the structure of the bosonic string model which represents a particular ($a = \sqrt{3}$) Melvin flux tube background in $D = 4$.

The corresponding type II superstring theory will be solved in Section 3 using RNS formulation. Its quantum Hamiltonian will be the free superstring one plus terms linear and quadratic in angular momentum operators. As a result, the mass spectrum can be explicitly determined.

The basic properties of the spectrum will be studied in Section 4. We will show that supersymmetry is broken and that there exist intervals of values of moduli parameters (Kaluza-Klein radius and magnetic field strength) for which the model is unstable. We shall also discuss the heterotic version of the model.

In Section 5 we shall consider the light-cone Green-Schwarz formulation of the theory, which turns out to be very simple. The breaking of supersymmetry will be related to the absence of Killing spinors in the Melvin background. We shall also compute the expression for the partition function on the torus which will be finite or infinite depending on the values of the parameters.

In Section 6 the results of Sections 3-5, obtained for the $a = \sqrt{3}$ Melvin model, will be generalized to a class of static magnetic flux tube models which includes, in particular, the $a = 1$ Melvin model. We shall explain the reason for solvability of these models and clarify the nature of perturbative instabilities that appear for generic values of the magnetic field parameters.

Section 7 will contain a summary and remarks on some generalizations. In particular, we shall comment on the relation between the $a = \sqrt{3}$ Melvin model and the superstring compactifications on twisted tori where supersymmetry is broken by discrete twist angles (or by the ‘Scherk-Schwarz’ mechanism).

¹ In addition to this perturbative instability there may be other instabilities of non-perturbative origin, discussed in the field-theory framework, in [5].

2. Bosonic string model for $a = \sqrt{3}$ Melvin background

In this and the following two sections we shall consider the supersymmetric version of the simplest representative in the class of static magnetic flux tube models of ref. [11] – the ‘Kaluza-Klein’ (or $a = \sqrt{3}$) Melvin model. It has properties similar to those of the more general models but yet can be solved in a rather simple way. This theory is special in that the corresponding σ -model has flat target space of non-trivial topology (other models in [11] have curved target spaces but are related to flat models by angular duality and globally non-trivial coordinate shifts). The relation of $a = \sqrt{3}$ Melvin background [1,3] to flat $D = 5$ theory was pointed out at the field-theory level in [4] (see also [5]) and at the string-theory level in [11,13].² The bosonic string model is defined by the following Lagrangian

$$L = L_0 + L_1 , \quad L_0 = -\partial_a t \partial^a t + \partial_a x_\alpha \partial^a x^\alpha , \quad (2.1)$$

$$L_1 = \partial_a \rho \partial^a \rho + \rho^2 (\partial_a \varphi + q \partial_a y) (\partial^a \varphi + q \partial^a y) + \partial_a y \partial^a y . \quad (2.2)$$

Here $\rho \geq 0$ and $0 < \varphi \leq 2\pi$ correspond to cylindrical coordinates on a (x_1, x_2) -plane, y is a circular ‘Kaluza-Klein’ coordinate with period $2\pi R$, and x^α include the flat x^3 -coordinate of $D = 4$ space-time and, e.g., 21 (or 5 in the superstring case) internal coordinates compactified on a torus.

The constant q plays the role of the magnetic field strength parameter in the 4-dimensional interpretation. L_1 can be represented in the ‘Kaluza-Klein’ form

$$L_1 = \partial_a \rho \partial^a \rho + F(\rho) \rho^2 \partial_a \varphi \partial^a \varphi + e^{2\sigma} (\partial_a y + \mathcal{A}_\varphi \partial_a \varphi) (\partial^a y + \mathcal{A}_\varphi \partial^a \varphi) , \quad (2.3)$$

so that the $D = 4$ background (metric, Abelian vector field \mathcal{A} and scalar σ) corresponding to (2.1) is indeed the $a = \sqrt{3}$ Melvin geometry

$$ds_4^2 = -dt^2 + d\rho^2 + F(\rho) \rho^2 d\varphi^2 + dx_3^2 , \quad (2.4)$$

$$\mathcal{A}_\varphi = qF(\rho)\rho^2 , \quad e^{2\sigma} = F^{-1}(\rho) , \quad F \equiv \frac{1}{1 + q^2 \rho^2} . \quad (2.5)$$

The non-trivial 3-dimensional part (2.2) of (2.1) is non-chiral (there is no antisymmetric tensor background) and the dilaton is constant. The 3-metric

$$ds^2 = d\rho^2 + \rho^2 (d\varphi + q dy)^2 + dy^2 , \quad (2.6)$$

² The string model corresponding to the $a = 1$ Melvin background [3] was constructed in [10] and solved in [11]. In contrast to the flat 3-space geometry of the $a = \sqrt{3}$ model (see below), in the $a = 1$ case the 3-space is curved and the (ρ, φ) -surface asymptotically closes at large ρ .

is flat (so that the model is automatically conformal to all orders) since locally one may introduce the coordinate $\theta = \varphi + qy$ and decouple y from ρ, φ . The global structure of this 3-space is non-trivial: the fixed ρ section is a 2-torus (with ρ -dependent conformal factor and complex modulus) which degenerates into a circle at $\rho = 0$. The space is actually regular everywhere, including $\rho = 0$ (this can easily be seen by rewriting (2.6) in terms of Cartesian coordinates of the 2-plane and y , cf. (2.7) below). It can also be obtained by factorizing R^3 over the group generated by translations in two angular directions: in the coordinates where $ds^2 = d\rho^2 + \rho^2 d\theta^2 + dy^2$ ($\theta = \varphi + qy$) one should identify the points $(\rho, \theta, y) = (\rho, \theta + 2\pi n + 2\pi qRm, y + 2\pi Rm)$ ($n, m = \text{integers}$), i.e. combine the shift by $2\pi R$ in y with a rotation by an arbitrary angle $2\pi qR$ in the 2-plane.³ Although the space is flat, the corresponding string theory will be non-trivial (already at the classical level due to the existence of winding string states and at the quantum level in the non-winding sector where there will be a ‘magnetic’ coupling to the total angular momentum in the 2-plane), representing an example of a gravitational Aharonov-Bohm-type phenomenon: the value of the parameter q does not influence the (zero) curvature of the space but affects the global properties like masses of string states.⁴

The new coordinate θ is globally defined (2π periodic) only for special integer periods of qy , i.e. for $qR = n$, $n = 0, \pm 1, \dots$. In these cases (2.3) is trivial, i.e. equivalent to a free bosonic string theory compactified on a circle.⁵ Models with $n < qR < n+1$ are equivalent to models with $0 < qR < 1$. We shall see that this periodicity condition in qR will be modified in superstring theory: because of the presence of fermions of half-integer spin, n will be replaced by $2n$, i.e. only models with $qR = 2n$ will be trivial. More generally, superstring theories with (R, q) and $(R, q + 2nR^{-1})$ will be equivalent.

It should be noted that it is $e^\sigma R$ in (2.3), (2.5) that plays the role of an effective Kaluza-Klein radius of the compact fifth dimension. Since e^σ grows with radial distance from the flux tube, it may seem that the Kaluza-Klein interpretation eventually breaks down (as was discussed in the field-theory context in [5] the standard Kaluza-Klein interpretation

³ Since the orbits of this group are non-compact (in contrast to, e.g., the case of a special 2-cone = R^2/Z_N orbifold [14,15]) the corresponding string model can be defined for arbitrary continuous values of the moduli parameters q, R .

⁴ Let us also note that the σ -model which is φ -dual to L_1 is a special case of models in [11], $\tilde{L}_1 = \partial_a \rho \partial^a \rho + \rho^{-2} \partial_a \tilde{\varphi} \partial^a \tilde{\varphi} + q\epsilon^{ab} \partial_a y \partial_b \tilde{\varphi} + \partial_a y \partial^a y + \alpha' R^{(2)}(\phi_0 - \ln \rho)$. The constant torsion term corresponds in $D = 4$ to the ‘Aharonov-Bohm’ gauge potential $B_{iy} \equiv \mathcal{B}_i = q\epsilon_{ij} x^j / x^2$, $F(\mathcal{B})_{ij} = -2\pi q\epsilon_{ij} \delta^{(2)}(x)$.

⁵ The Kaluza-Klein field theory is also trivial in this case, since the corresponding solution of $D = 5$ Einstein theory is equivalent to (Minkowski 4-space) $\times S^1$ (any ‘observable’ computed in terms of the 4-dimensional variables, i.e. on the background (2.4), (2.5), should have the same value as in the $D = 5$ theory).

would apply only for $qR \ll 1$). As for the higher dimensional string theory, it is defined for an arbitrary q . Its mass spectrum derived below will not contain extra light states (in the non-winding sector) for small R (and arbitrary q). Still, its 4-dimensional ‘magnetic’ interpretation will directly apply only for small qR .

The model (2.1), (2.2) has a straightforward generalization where φ is ‘mixed’ with several internal coordinates: $\partial_a \varphi + q \partial_a y \rightarrow \partial_a \varphi + q_r \partial_a y^r$, etc. The corresponding $D = 4$ background contains several magnetic fields and moduli fields.

It is useful to represent (2.3) in the following equivalent form, introducing $x = x_1 + ix_2 = \rho e^{i\varphi}$:

$$L_1 = (\partial_a x_i - q \epsilon_{ij} x_j \partial_a y)(\partial^a x_i - q \epsilon_{ij} x_j \partial^a y) + \partial_a y \partial^a y \quad (2.7)$$

$$\begin{aligned} &= (\partial_a x + i q x \partial_a y)(\partial^a x^* - i q x^* \partial^a y) + \partial_a y \partial^a y \\ &= D^a x D_a^* x^* + \partial_a y \partial^a y, \quad D_a \equiv \partial_a + i q \partial_a y; \end{aligned} \quad (2.8)$$

we therefore get a charged complex 2d scalar field x in a flat 2d gauge potential. Since y is compact, the effect of this gauge potential will be non-trivial.

The conformal theory corresponding to the bosonic model (2.1) was solved in [11] (as a special case of a more general class of models) by observing that the solution of the classical string equations can be expressed in terms of free fields and applying the canonical quantization. This procedure is particularly simple in the present model (2.7). Since the ‘ $U(1)$ potential’ $q \partial_a y$ in (2.8) is flat, x can be formally ‘rotated’ to decouple x from y . Then y satisfies the free field equation and x is also expressed in terms of free fields. The only interaction which effectively survives in the final expressions is the coupling of x to the derivative of the zero mode part of y , $y_* = y_0 + 2\alpha' p\tau + 2Rw\sigma$. It is then straightforward to carry out the canonical quantization procedure, expressing all observables in terms of free oscillators. The resulting Hamiltonian is given by the sum of the free string Hamiltonian plus $O(q)$ and $O(q^2)$ terms depending on the left and right components of the free string angular momentum operators \hat{J}_L and \hat{J}_R [11].

As was shown in [11], this bosonic string model is stable in the non-winding sector, where there are no new instabilities in addition to the usual flat space tachyon. This means, in particular, that the Kaluza-Klein field theory corresponding to the Melvin background is perturbatively stable with respect to the ‘massless’ (graviton, vector, scalar) *and* massive perturbations. This theory may still be unstable at a non-perturbative level [5]. At the same time, there exists a range of parameters q and R for which there are tachyonic states in the winding sector, i.e. this string model is unstable against certain winding-mode perturbations.

This instability (whose origin is essentially in the gyromagnetic coupling term $wqR(\hat{J}_R - \hat{J}_L)$, which may have negative sign) is not related to the presence of the flat bosonic string tachyon and may thus be expected to survive (for certain values of q and R) also in the superstring case. This, indeed, is what will be found below.

3. Solution of the superstring Melvin model

In what follows we shall consider the type II superstring version of (2.1) (heterotic models with the magnetic field in the Kaluza-Klein sector can be obtained by straightforward ‘left’ or ‘right’ truncations and have similar properties). Since (in contrast to the constant magnetic field model in [9,12]) the ∂y -dependent interaction terms in (2.2) are non-chiral, there does not exist an associated heterotic string model with the magnetic field embedded in the internal gauge sector.

In this section we shall consider the RNS formulation of the model. The model can be solved also by using directly the Green-Schwarz [16,17] formulation (see Section 5), which confirms (and clarifies certain aspects of) the RNS solution. The (1,1) world-sheet supersymmetric extension of the model (2.2),(2.8) has the form ($x^\mu \equiv (x^i, y)$)

$$L_{\text{RNS}} = G_{\mu\nu}(x)\partial_+x^\mu\partial_-x^\nu + \lambda_{Rm}(\delta_n^m\partial_+ + \omega_{n\mu}^m\partial_+x^\mu)\lambda_R^n + \lambda_{Lm}(\delta_n^m\partial_- + \omega_{n\mu}^m\partial_-x^\mu)\lambda_L^n . \quad (3.1)$$

$\lambda^m = e_\mu^m\lambda^\mu$ are vierbein components of the 2d Majorana-Weyl spinors and $\omega_{n\mu}^m$ is the (flat) spin connection. There are no quartic fermionic terms since the metric is flat. In the natural basis $e^i = dx^i - q\epsilon^{ij}x_j dy$, $e^y = dy$, the spin connection 1-form has the following components

$$\omega^{ij} = -q\epsilon^{ij}dy , \quad \omega^{iy} = 0. \quad (3.2)$$

In terms of the left and right Weyl spinors $\lambda = \lambda_1 + i\lambda_2$ corresponding to $x = x_1 + ix_2$ and $\lambda^y \equiv \psi$, we get (cf. (2.8))

$$L_{\text{RNS}} = D_+x D_-^*x^* + \partial_+y\partial_-y + \lambda_R^*D_+\lambda_R + \lambda_L^*D_-\lambda_L + \psi_R\partial_+\psi_R + \psi_L\partial_-\psi_L , \quad D_\pm \equiv \partial_\pm + iq\partial_\pm y , \quad (3.3)$$

where the covariant derivative D_\pm is the same as in (2.8), i.e. it contains the flat $U(1)$ potential. This means that, as in the bosonic case, it is possible to redefine the fields x, λ so that the only non-trivial coupling that will remain at the end will be to the zero mode of y .⁶ Although it may seem that, as in the bosonic case, the model with $qR = n$ should be equivalent to the free superstring theory compactified on a circle (since for $qR = n$ one can, in principle, eliminate the coupling terms in (3.3) by rotating the fields) this will not actually be true unless the integer n is even, $n = 2k$. The non-triviality for $n = 2k + 1$ is

⁶ One can directly generalize the bosonic case discussion by replacing x^i, y by (1,1) superfields and observing that the zero mode part of the y -superfield can have only the bosonic component y_* . Note that both the momentum and the winding parts of y_* are on an equal footing in (3.3): the model is non-trivial (not equivalent to the free string one) already in the non-winding sector.

directly related to the presence of space-time fermions in the spectrum, which change sign under 2π spatial rotation accompanying the periodic shift in y (see below and Section 5).

Taking the world-sheet to be a cylinder (τ, σ) ($0 < \sigma \leq \pi$) we can solve the classical equations corresponding to (3.3) by introducing the fields X and $\Lambda_{R,L}$, which will satisfy the free string equations but will have ‘twisted’ boundary conditions $(\sigma_{\pm} \equiv \tau \pm \sigma)^7$

$$x(\tau, \sigma) = e^{-iqy(\tau, \sigma)} X(\tau, \sigma) , \quad \partial_+ \partial_- X = 0 , \quad X = X_+(\sigma_+) + X_-(\sigma_-) , \quad (3.4)$$

$$X(\tau, \sigma + \pi) = e^{2\pi i \gamma} X(\tau, \sigma) , \quad \gamma \equiv qRw , \quad (3.5)$$

$$\lambda_{R,L}(\tau, \sigma) = e^{-iqy(\tau, \sigma)} \Lambda_{R,L}(\tau, \sigma) , \quad \partial_{\pm} \Lambda_{R,L} = 0 , \quad \Lambda_{R,L} = \Lambda_{R,L}(\sigma_{\mp}) , \quad (3.6)$$

$$\Lambda_{R,L}(\tau, \sigma + \pi) = \pm e^{2\pi i \gamma} \Lambda_{R,L}(\tau, \sigma) , \quad (3.7)$$

with the signs ‘ \pm ’ in (3.7) corresponding to the Ramond (R) and Neveu-Schwarz (NS) sectors. The crucial observation is that y still satisfies the free-field equation:

$$\partial_+ \partial_- y = 0 , \quad y = y_* + y' , \quad y_* = y_0 + p_+ \sigma_+ + p_- \sigma_- . \quad (3.8)$$

We have used the fact that the fields x, y, λ must obey the usual closed-string boundary conditions,

$$x(\tau, \sigma + \pi) = x(\tau, \sigma) , \quad y(\tau, \sigma + \pi) = y(\tau, \sigma) + 2\pi R w , \quad w = 0, \pm 1, \dots , \quad (3.9)$$

$$\lambda_{R,L}(\tau, \sigma + \pi) = \pm \lambda_{R,L}(\tau, \sigma) . \quad (3.10)$$

The explicit expressions for the fields $X = X_+ + X_-$ and $\Lambda_{L,R}$ are then

$$X_{\pm}(\sigma_{\pm}) = e^{\pm 2i\gamma\sigma_{\pm}} \mathcal{X}_{\pm}(\sigma_{\pm}) , \quad \mathcal{X}_{\pm}(\sigma_{\pm} \pm \pi) = \mathcal{X}_{\pm}(\sigma_{\pm}) , \quad (3.11)$$

$$\Lambda_{L,R}(\sigma_{\pm}) = e^{\pm 2i\gamma\sigma_{\pm}} \eta_{L,R}(\sigma_{\pm}) , \quad (3.12)$$

where \mathcal{X}_{\pm} and $\eta_{L,R}$ are the free fields with the standard free closed string boundary conditions, i.e.

$$\mathcal{X}_+ = i\sqrt{\frac{1}{2}\alpha'} \sum_{n \in \mathbf{Z}} \tilde{a}_n e^{-2in\sigma_+} , \quad \mathcal{X}_- = i\sqrt{\frac{1}{2}\alpha'} \sum_{n \in \mathbf{Z}} a_n e^{-2in\sigma_-} , \quad (3.13)$$

$$\eta_R^{(\text{NS})} = \sqrt{2\alpha'} \sum_{r \in \mathbf{Z} + \frac{1}{2}} c_r e^{-2ir\sigma_-} , \quad \eta_R^{(\text{R})} = \sqrt{2\alpha'} \sum_{n \in \mathbf{Z}} d_n e^{-2in\sigma_-} , \quad (3.14)$$

⁷ The twist parameter γ can be interpreted as a flux corresponding to the 2d $U(1)$ field $A_a = q\partial_a y$ on the cylinder, $\int A = 2qRw \int d\sigma = 2\pi\gamma$. Note that we have redefined γ by factor of 2 compared to our previous papers [11,12].

and similar expressions for the left fermions with oscillators having extra tildes. We can then proceed with canonical quantization of the model expressing the observables in terms of the above free oscillators. It is convenient to choose the light-cone gauge, eliminating the oscillator part of $u = y - t$ (see [11,12] for details). Then

$$u = u_* \equiv u_0 + 2\alpha'(p + E)\tau + 2Rw\sigma , \quad (3.15)$$

$$p = p_y - q\hat{J} , \quad p_y = mR^{-1} , \quad m = 0, \pm 1, \dots ,$$

where E is the total energy, m is the Kaluza-Klein linear momentum number, w is the winding number and $\hat{J} = \hat{J}_R + \hat{J}_L$ is the total angular momentum in the 2-plane.

In what follows we shall first assume that w (or qR) is such that $0 \leq \gamma < 1$ and then consider generalizations to other values of $\gamma = qRw$. The angular momentum operators that appear in the final Hamiltonian contain the orbital momentum parts plus the spin parts (with the latter having the standard free superstring form [17])

$$\hat{J}_R = -b_0^\dagger b_0 - \frac{1}{2} + \sum_{n=1}^{\infty} (b_{n+}^\dagger b_{n+} - b_{n-}^\dagger b_{n-}) + \hat{K}_R , \quad (3.16)$$

$$\hat{K}_R^{(\text{NS})} = - \sum_{r=\frac{1}{2}}^{\infty} (c_r^* c_r + c_{-r} c_{-r}^*), \quad \hat{K}_R^{(\text{R})} = -[d_0^*, d_0] + \sum_{n=1}^{\infty} (d_n^* d_n + d_{-n} d_{-n}^*).$$

The expression of \hat{J}_L is similar, with the reversed sign of the orbital momentum terms. Here b 's are the free creation and annihilation operators related to the modes in (3.13) by rescaling by factors $(n \pm \gamma)^{1/2}$, see [11]. The eigenvalues of $\hat{J}_{L,R}$ are

$$\hat{J}_{L,R} = \pm(l_{L,R} + \frac{1}{2}) + S_{L,R} , \quad \hat{J} \equiv \hat{J}_L + \hat{J}_R = l_L - l_R + S_L + S_R, \quad (3.17)$$

where the orbital momenta $l_{L,R} = 0, 1, 2, \dots$ (which replace the continuous linear momenta p_1, p_2 in the 2-plane for non-zero values of γ) are the analogues of the Landau quantum number and $S_{R,L}$ are the spin components.⁸

The number of states operators \hat{N}_R and \hat{N}_L have the standard form

$$\hat{N}_{R,L} = N_{R,L} - a , \quad a^{(\text{R})} = 0 , \quad a^{(\text{NS})} = \frac{1}{2} , \quad (3.18)$$

⁸ In the case $\gamma = 0$ (or, more generally, $\gamma = n$) the zero-mode structure changes in that the translational invariance in the 2-plane is recovered, see [9,11]. This leads to a slight modification in the formulas (the operators $b_0^\dagger, b_0, \tilde{b}_0^\dagger, \tilde{b}_0$ in the expressions below are then replaced by standard zero-mode operators $x_{1,2}, p_{1,2}$). We shall not explicitly indicate this in what follows.

where, e.g. in the Ramond sector,

$$N_R^{(R)} = \sum_{n=1}^{\infty} n (b_{n+}^\dagger b_{n+} + b_{n-}^\dagger b_{n-} + b_{n\alpha}^\dagger b_{n\alpha} + d_n^* d_n + d_{-n} d_{-n}^* + d_{-n\alpha} d_{n\alpha}) . \quad (3.19)$$

$N_L^{(R)}$ has a similar expression in terms of operators with tildes (there are no contributions with oscillators corresponding to y and t since we used the light-cone gauge). Under the usual GSO projection (which is necessary for the correspondence with the Green-Schwarz formulation and with the free RNS superstring theory in the limit $q = 0$ but will not imply the space-time supersymmetry in the present case) \hat{N}_R and \hat{N}_L can take only non-negative integer values (and correspond to the number of states operators of the light-cone Green-Schwarz formulation).

The resulting expressions for the Hamiltonian and level matching constraint are⁹

$$\hat{H} = \frac{1}{2}\alpha' (-E^2 + p_\alpha^2 + \frac{1}{2}Q_L^2 + \frac{1}{2}Q_R^2) + \hat{N}_R + \hat{N}_L \quad (3.20)$$

$$- \alpha' q (Q_L \hat{J}_R + Q_R \hat{J}_L) + \frac{1}{2}\alpha' q^2 \hat{J}^2 ,$$

$$Q_{L,R} \equiv \frac{m}{R} \pm \frac{wR}{\alpha'} , \quad (3.21)$$

$$\hat{N}_R - \hat{N}_L = mw . \quad (3.22)$$

\hat{H} can be represented also in the following (‘free superstring compactified on a circle’) form which clarifies its structure and is useful for generalizations

$$\hat{H} = \frac{1}{2}\alpha' (-E^2 + p_\alpha^2 + m'^2 R^{-2} + \alpha'^{-2} w^2 R^2) + \hat{N}'_R + \hat{N}'_L , \quad (3.23)$$

where $m', \hat{N}'_R, \hat{N}'_L$ (which are no longer integer in general) are defined by

$$m' \equiv m - qR\hat{J} , \quad (3.24)$$

$$\hat{N}'_R \equiv \hat{N}_R - \gamma \hat{J}_R , \quad \hat{N}'_L \equiv \hat{N}_L + \gamma \hat{J}_L , \quad \gamma = qRw . \quad (3.25)$$

Up to the orbital momentum terms, $\hat{N}'_{R,L}$ can be put into the same form as free operators $\hat{N}_{R,L}$ (see (3.19)) with the factor n replaced by $n \pm \gamma$.

⁹ Symmetrizing the classical expressions for the Virasoro operators L_0, \tilde{L}_0 , we then normal-order them and use the generalized ζ -function prescription. In contrast to the bosonic case [11] here the γ^2 normal ordering terms cancel out between bosons and fermions. For example, in the NS-sector one obtains: $L_0 \rightarrow L_0 - \frac{1}{2}(1 - \gamma)$. The latter normal-ordering constants are naturally absorbed into $\hat{N}_{R,L}$ and $\hat{J}_{R,L}$.

The Virasoro condition $\hat{H} = 0$ leads to the following mass spectrum

$$M^2 \equiv E^2 - p_\alpha^2 = M_0^2 - 2qR^{-1}m\hat{J} - 2\alpha'^{-1}qRw(\hat{J}_R - \hat{J}_L) + q^2\hat{J}^2, \quad (3.26)$$

where M_0 is the mass operator of the free superstring compactified on a circle (for simplicity we ignore the contributions of the other 5 free compactified dimensions, i.e. the corresponding momenta are set equal to zero)

$$M_0^2 = 2\alpha'^{-1}(\hat{N}_L + \hat{N}_R) + m^2R^{-2} + \alpha'^{-2}w^2R^2. \quad (3.27)$$

The equivalent form of (3.26), which demonstrates that in general M^2 is not positive-definite in the winding sector, is

$$\begin{aligned} M^2 &= 2\alpha'^{-1}(\hat{N}_R + \hat{N}_L) + (mR^{-1} - q\hat{J})^2 \\ &+ [\alpha'^{-1}wR - q(\hat{J}_R - \hat{J}_L)]^2 - q^2(\hat{J}_R - \hat{J}_L)^2. \end{aligned} \quad (3.28)$$

Let us consider first the zero winding sector $w = 0$ ($\gamma = 0$) where $M^2 = 2\alpha'^{-1}(\hat{N}_R + \hat{N}_L) + m'^2R^{-2}$. It is clear from (3.24) that the mass spectrum is then invariant under

$$q \rightarrow q + 2nR^{-1}, \quad n = 0, \pm 1, \dots, \quad (3.29)$$

since this transformation can be compensated by $m \rightarrow m - 2n\hat{J} = \text{integer}$. Note that since \hat{J} can take both integer (NS-NS, R-R sectors) and *half-integer* (NS-R, R-NS sectors) values, the symmetry of the bosonic part of the spectrum $q \rightarrow q + nR^{-1}$ is *not* a symmetry of its fermionic part, i.e. the full superstring spectrum is invariant only under (3.29).

The same conclusion about the periodicity in q is true in general for $w \neq 0$. In the form given above, eqs. (3.20), (3.26), (3.28) are valid for $0 \leq w < (qR)^{-1}$, i.e. for $0 \leq \gamma < 1$. The generalization to other values of γ , e.g. γ in any interval $n \leq \gamma < n + 1$, $n = \text{integer}$, is straightforward (see also [11]). The net effect is the replacement of γ in (3.25) by $\gamma - n$, i.e. qRw in (3.26) by $qRw - n$. The general form of the mass operator is thus

$$\begin{aligned} \alpha' M^2 &= 2(\hat{N}_R + \hat{N}_L) + \alpha'(mR^{-1} - q\hat{J})^2 + \alpha'^{-1}w^2R^2 \\ &- (\gamma - n)(\hat{J}_R - \hat{J}_L). \end{aligned} \quad (3.30)$$

As will be clear from a comparison with the Green-Schwarz formulation, one should use the standard GSO projection for $2k \leq \gamma < 2k + 1$, and the ‘reversed’ one for $2k + 1 \leq \gamma < 2k + 2$, ($k = 0, \pm 1, \dots$). The ‘reversal’ of GSO for $2k + 1 \leq \gamma < 2k + 2$ implies that in this interval only states having half-integer eigenvalues of the operators $\hat{N}_{L,R}$ will survive, including, in particular, scalar odd-winding tachyon states with $\hat{N}_L = \hat{N}_R = -\frac{1}{2}$. This prescription (which appeared also in the model of [18], see also [19,20], related to the special case

$qR = 2n + 1$ of our model) is consistent with the modular invariance of the partition function (see Section 5).

For fixed radius R the mass spectrum is thus periodic in q , i.e. it is mapped into itself under (3.29) (combined with $m \rightarrow m - 2n\hat{J}$). In the case $qR = 2n$ (i.e. $\gamma = 2nw = 2k$) the spectrum is thus equivalent to the standard spectrum of the free superstring theory compactified on a circle. For $qR = 2n + 1$ (i.e. $\gamma = (2n + 1)w = 2k + 1$ if w is odd) the spectrum is the same as that of free superstring compactified on a circle with antiperiodic boundary conditions for space-time fermions [18] (see also [19,20]). This relation will become clear in the Green-Schwarz formulation (Section 5). In particular, it will be apparent that the interaction term in the superstring action can be eliminated by a globally defined field transformation only if $qR = 2n$, while for $qR = 2n + 1$ this can be done at the expense of imposing antiperiodic boundary conditions (in the y -direction) on fermions (under the rotation by the angle $2\pi qR = 2\pi$ in the 2-plane, which is associated with a periodic shift in y , the bosons remain invariant but the spinors change sign).

Let us note that, in contrast to the case of the free string compactified on a circle, the mass spectrum (3.28) (for generic qR) is *not* invariant under the naive duality transformation $R \rightarrow \alpha' R^{-1}$ (accompanied by some redefinition of quantum numbers such as $(w, m) \rightarrow (m, w)$ in the free string case). Unlike, e.g. the free string or $a = 1$ Melvin model [10,11], the action (2.2) does not preserve its form under the duality transformation in y , i.e. the y -duality maps (2.2) into a *different* σ -model (belonging to the 3-parameter class of models in [11])

$$\begin{aligned} \tilde{L} = & \partial_+ \rho \partial_- \rho + F(\rho) \rho^2 (\partial_+ \varphi + q \partial_+ \tilde{y}) (\partial_- \varphi - q \partial_- \tilde{y}) + \partial_+ \tilde{y} \partial_- \tilde{y} \\ & + \mathcal{R}(\phi_0 + \tfrac{1}{2} \ln F), \quad F \equiv (1 + q^2 \rho^2)^{-1}, \quad \mathcal{R} \equiv \tfrac{1}{4} \alpha' \sqrt{g} R^{(2)}. \end{aligned} \quad (3.31)$$

This model is equivalent to (2.2) at the CFT level, i.e. it has, in particular, the same mass spectrum (3.28).

4. Mass spectrum: supersymmetry breaking and (in)stability

There are two immediate consequences that can be drawn from the above expressions (3.26),(3.28) for M^2 :

- (i) the space-time supersymmetry is broken for $qR \neq 2n$;
- (ii) there exists a range of values of parameters q and R for which there are tachyonic states in the spectrum.

Suppose that we start with the free superstring compactified on a circle y and study what happens with the spectrum when we switch on the magnetic field, $q \neq 0$. Since the mass shift in (3.26) involves *both* components \hat{J}_L and \hat{J}_R of the angular momentum (with independent generically non-vanishing coefficients), it is easy to see that the masses

of bosons and fermions that were equal for $q = 0$ will become different for $q \neq 0$. Indeed, it is impossible to have both \hat{J}_L and \hat{J}_R equal for bosons and fermions.¹⁰

Supersymmetry is absent already in the non-winding sector (where the coupling is to the total angular momentum \hat{J}). For example, the free superstring massless (ground) states ($\hat{N}_{L,R} = 0 = m = w$) will, according to (3.26), get masses $M = |q\hat{J}|$ proportional to their total angular momenta, which must be integer for bosons and half-integer for fermions (cf. (3.17)). Note that these states are neutral, so that from the 4-dimensional point of view the shift in the masses can be interpreted as a gravitational effect. This shift implies, in particular, that supersymmetry is broken at the field-theory (e.g., $D = 5$ supergravity) level, in agreement with the absence of Killing spinors in the Melvin background (see Section 5).

In the absence of supersymmetry some instabilities of the bosonic string model may survive also in the superstring case. As in the bosonic case, the mass operator (3.28) is positive in the non-winding sector, but tachyonic states may appear in the winding sector (we use the name ‘tachyon’ for a state with $M^2 < 0$; it should be remembered, of course, that the string states we are discussing propagate in curved $D = 4$ space-time). Consider, for example, the NS-NS superstring winding states with zero Kaluza-Klein momentum and zero orbital momentum quantum numbers and with maximal absolute values of spins $S_{R,L}$ at given levels

$$w > 0, \quad m = 0, \quad l_R = l_L = 0, \quad S_R = \hat{N}_R + 1, \quad S_L = -\hat{N}_L - 1. \quad (4.1)$$

We will restrict our consideration to states for which $0 < qRw < 1$ (states with $w > (qR)^{-1}$ can be analysed in a similar way, see (3.30)). Then (3.22),(3.26) imply

$$\hat{N}_R = \hat{N}_L \equiv N, \quad \hat{J} = 0, \quad \hat{J}_R - \hat{J}_L = 2N + 1, \quad (4.2)$$

$$\alpha' M^2 = 4N + \alpha'^{-1} w^2 R^2 - 2qRw(2N + 1). \quad (4.3)$$

A state with given N and w will be tachyonic for $q > q_{\text{cr}}$,

$$q_{\text{cr}} = \frac{4N + \alpha'^{-1} w^2 R^2}{2(2N + 1)wR}. \quad (4.4)$$

For $N = 0$ we get $\alpha' q_{\text{cr}} = \frac{1}{2} wR$. The condition $qRw < 1$ is satisfied provided $wR < \sqrt{2\alpha'}$.

¹⁰ In the constant magnetic model [12] where the coupling to the magnetic field was only through \hat{J}_R (half of) the supersymmetry was preserved in the type II superstring and in the ‘left-right’ symmetric and ‘left’ heterotic models. There \hat{J}_R (and thus the mass shift) was the same for bosons and fermions.

In general, it is easy to check (using the fact that $-\hat{N}_{R,L} - 1 \leq S_{R,L} \leq \hat{N}_{R,L} + 1$) that states with $M^2 < 0$ can be present only for $R < \sqrt{2\alpha'}$, i.e. the full spectrum is tachyon-free if $R > \sqrt{2\alpha'}$. For fixed $R < \sqrt{2\alpha'}$ the minimal value of the magnetic field strength parameter at which tachyons first appear is $\alpha' q_{\text{cr}} = \frac{1}{2}R$, corresponding to the $N = 0, w = 1$ case of (4.4). Tachyons with $N = 1$ are found at larger values of the magnetic field $q > q_{\text{cr}}$ with q_{cr} given by (4.4) for $N = 1$, etc.

All other sectors (R-R, R-NS, NS-R) are tachyon-free. Let us consider, for example, the fermionic states of the R-NS -sector with the following quantum numbers (cf. (4.1)): $w > 0$, $l_R = l_L = 0$, $S_L = -\hat{N}_L - 1$, $S_R = \hat{N}_R + \frac{1}{2}$. For $qRw < 1$ the corresponding mass formula is (cf. (4.3))

$$\begin{aligned} \alpha' M^2 = & 2(\hat{N}_R + \hat{N}_L)(1 - qRw) + \alpha'^{-1}(wR - \frac{1}{2}\alpha'q)^2 \\ & + \alpha'R^{-2}m(1 - qRw)[m(1 - qRw) + qR] , \end{aligned} \quad (4.5)$$

i.e. M^2 is non-negative. The winding fermionic state with $\hat{N}_{R,L} = 0 = m$, $w = 1$ thus becomes massless at $q = 2R/\alpha'$.

Some values of the radius, such as $R = \sqrt{2\alpha'}$, are special. For $R = \sqrt{2\alpha'}$ the value of M^2 is non-negative. As the magnetic field q is gradually increased from zero, the masses of the infinite number of modes belonging to the set (4.1) with $w = 1$ will decrease. They will simultaneously approach zero when q will approach $R^{-1} = 1/\sqrt{2\alpha'}$ ($qRw \rightarrow 1$). At this point there is a discontinuity in M^2 since, for $qR = 1$, the spectrum is equivalent to that of a free superstring on a circle with antiperiodic boundary conditions for fermions (see the discussion on the periodicity of M^2 in q in the previous section).

The structure of the spectrum of the present model is thus different from that of the constant magnetic field model [9,12] in which infinitely many instabilities appeared for any arbitrarily small value of the magnetic field.

One can consider also the heterotic version of the above model (where the magnetic field is embedded in the Kaluza-Klein sector) by combining the ‘left’ or ‘right’ part of the superstring model with the free internal part. The mass formula and the level matching condition in this case take the following form (cf. (3.28),(3.22))

$$\begin{aligned} \alpha' M^2 = & 2(\hat{N}_R + \hat{N}_L + \frac{1}{2}p_I^2) + \alpha'(mR^{-1} - q\hat{J})^2 \\ & + \alpha'^{-1}(wR)^2 - 2qRw(\hat{J}_R - \hat{J}_L), \end{aligned} \quad (4.6)$$

$$\hat{N}_R - \hat{N}_L = mw + \frac{1}{2}p_I^2, \quad \hat{N}_R = 0, 1, 2, \dots, \quad \hat{N}_L = N_L - 1 = -1, 0, 1, \dots,$$

where \hat{N}_L contains only the free internal oscillator modes (see [12] for notation). There are instabilities similar to the ones discussed in the above type II model. In addition, there are other instabilities which, in the case of the special ‘self-dual’ value of the radius $R = \sqrt{\alpha'}$, appear for infinitesimal values of the magnetic field. These are just the usual Yang-Mills-type magnetic instabilities, associated with the gauge bosons (with quantum numbers $m = w = \pm 1$, $p_I^2 = l_R = l_L = 0$, $\hat{N}_R = N_L = 0$, $S_R = 1$, $S_L = 0$) of the $SU(2)_L$ group.

5. Green-Schwarz formulation and partition function

The supersymmetry breaking is related to the coupling of fermions in (3.3) to the flat but globally non-trivial $U(1)$ connection. This can be seen explicitly in the Green-Schwarz formulation [16,17] where the absence of supersymmetry is connected to the non-existence of Killing spinors in a given bosonic background. Let us consider the Killing spinor equation

$$(\partial_\mu + \frac{1}{4}\omega_\mu^{mn}\gamma_{mn})\epsilon = 0 , \quad (5.1)$$

in the $D = 3$ background corresponding to (2.7). Here $\epsilon = \epsilon(x^i, y)$ is a space-time spinor and ω_μ^{mn} is the same flat spin connection as in (3.1),(3.2) so that (5.1) reduces to

$$(\partial_y - \frac{1}{4}q\epsilon^{ij}\gamma_{ij})\epsilon = 0 . \quad (5.2)$$

The formal solution of (5.2)

$$\epsilon(y) = \exp(\frac{1}{4}q\epsilon^{ij}\gamma_{ij}y) \epsilon(0) , \quad (5.3)$$

does not, however, satisfy the periodic boundary condition in y , $\epsilon(y + 2\pi R) = \epsilon(y)$ (unless $qR = 2n$ when the Killing spinor does exist, in agreement with the fact that in this case the theory is equivalent to the free superstring).¹¹

The conclusion is that for $qR \neq 2n$ there is no residual space-time supersymmetry in the higher-dimensional (e.g., $D = 5$ supergravity) counterpart of the $a = \sqrt{3}$ Melvin background. The absence of Killing spinors in the case of the $a = 0$ Melvin solution of the Einstein-Maxwell theory was previously mentioned in [2].

Given a generic curved bosonic background, the corresponding Green-Schwarz (GS) superstring action [16] defines a complicated non-linear 2d theory (see, e.g., [23,24,25]). This theory appears to be more tractable when one is able to fix a light-cone gauge (in particular, when the background is flat at least in one time-like and one space-like directions). Then the action takes a ‘ σ -model’ form, which can be explicitly determined [24], e.g. by comparing with the known light-cone superstring vertex operators [17]. This light-cone

¹¹ Redefining $\varphi \rightarrow \varphi - qt$ (which is always possible since t is non-compact) one can put the Lagrangian (2.1),(2.7) in the ‘plane-wave’ form (see [11]) $L = \partial_a u \partial^a v + qx^i x_i \partial_a u \partial^a u + 2q\epsilon_{ij} x^i \partial_a x^j \partial^a u + \partial_a x_i \partial^a x^i$, $u = y - t$, $v = y + t$. Then the absence of supersymmetry in the Melvin model seems to contradict usual claims that plane-wave backgrounds are supersymmetric (see, e.g., [21,22]). In fact, there is no contradiction since the supersymmetry may be broken in the plane-wave backgrounds if the direction y in which the wave is propagating is compact. If the spin connection has constant y (or u) component, the corresponding Killing spinor equation may not have solutions consistent with periodic boundary conditions in the y -direction.

gauge action becomes very simple (quadratic in fermions) when the background geometry is flat as in the case of the Melvin model (2.2) (cf. (3.1))

$$L_{GS} = G_{\mu\nu}(x)\partial_+x^\mu\partial_-x^\nu + i\mathcal{S}_R\mathcal{D}_+\mathcal{S}_R + i\mathcal{S}_L\mathcal{D}_-\mathcal{S}_L, \quad (5.4)$$

$$\mathcal{D}_a \equiv \partial_a + \frac{1}{4}\omega_\mu^{mn}\gamma_{mn}\partial_ax^\mu.$$

Here $\mathcal{S}_{R,L}^p$ ($p = 1, \dots, 8$) are the right and left real spinors of $SO(8)$ (we consider type IIA theory). In the case of (2.2) we get (cf. (2.8),(3.3),(5.2))¹²

$$\begin{aligned} L_{GS} = & (\partial_+ + iq\partial_+y)x(\partial_- - iq\partial_-y)x^* + \partial_+y\partial_-y \\ & + i\mathcal{S}_R(\partial_+ - \frac{1}{4}q\epsilon^{ij}\gamma_{ij}\partial_+y)\mathcal{S}_R + i\mathcal{S}_L(\partial_- - \frac{1}{4}q\epsilon^{ij}\gamma_{ij}\partial_-y)\mathcal{S}_L. \end{aligned} \quad (5.5)$$

It is natural to decompose the $SO(8)$ spinors according to $SO(8) \rightarrow SU(4) \times U(1)$, i.e. $\mathcal{S}_L^p \rightarrow (\mathcal{S}_L^r, \bar{\mathcal{S}}_L^r)$, $\mathcal{S}_R^p \rightarrow (\mathcal{S}_R^r, \bar{\mathcal{S}}_R^r)$, $r = 1, \dots, 4$ ($\bar{\mathcal{S}}_{L,R}$ are complex conjugates of $\mathcal{S}_{L,R}$). With respect to the rotational group $U(1)$ of the plane, $\mathcal{S}_R^r, \bar{\mathcal{S}}_L^r$ and $\bar{\mathcal{S}}_R^r, \mathcal{S}_L^r$ have the charges $\frac{1}{2}$ and $-\frac{1}{2}$ (the bosonic fields x, x^* have the charges ± 1). Then the fermionic terms in (5.5) become

$$L_{GS}(\mathcal{S}_{L,R}) = i\bar{\mathcal{S}}_R^r(\partial_+ + \frac{1}{2}iq\partial_+y)\mathcal{S}_R^r + i\bar{\mathcal{S}}_L^r(\partial_- - \frac{1}{2}iq\partial_-y)\mathcal{S}_L^r. \quad (5.6)$$

The condition that the action (5.4),(5.5) has residual supersymmetry invariance $\mathcal{S} \rightarrow \mathcal{S} + \epsilon(x)$ is equivalent to $\mathcal{D}_a\epsilon(x(\tau, \sigma)) = \partial_ax^\mu(\partial_\mu + \frac{1}{4}\omega_\mu^{mn}\gamma_{mn})\epsilon(x) = 0$. The absence of supersymmetry invariance is the consequence of the absence of zero modes of the above covariant derivative operators, or, equivalently, of the non-existence of solutions of the Killing spinor equations (5.1), (5.2).

The connection terms in the covariant derivatives in the fermionic part of the GS action (5.6) have extra coefficients $\frac{1}{2}$ with respect to the ones in the RNS action (3.3). This immediately implies that the full theory is periodic under $qR \rightarrow qR + 2n$.

As in the bosonic and RNS cases, we can explicitly solve the classical string equations corresponding to (5.5) with the final result that the only essential difference, as compared to the free superstring case, is the coupling of bosons and fermions to the zero-mode part of the flat $U(1)$ connection ∂_ay_* . The expressions for the superstring Hamiltonian and mass spectrum are essentially the same as in (3.20), (3.30), where, for $2k \leq \gamma < 2k + 1$ the operators $\hat{N}_{L,R}, \hat{J}_{L,R}$ have the usual free GS superstring form, which is similar to their form in the R-sector of the RNS formalism with vanishing zero-point energy. For $2k - 1 \leq \gamma < 2k$ the operators $\hat{N}_{L,R}$ have the ‘NS-sector’ form, i.e. they take half-integer eigenvalues starting from $-\frac{1}{2}$.

¹² Light-cone gauge may be fixed on $t - x_3$. An equivalent approach is first to redefine the fields to eliminate the ‘oscillating’ part of y from the interaction terms and then impose the gauge on $u = t - y$.

The reason for this change from integer to half-integer eigenvalues can be understood directly from the action (5.6): the classical solution is (cf. (3.6),(3.7))

$$\mathcal{S}_{R,L}(\tau, \sigma) = e^{-\frac{i}{2}qy(\tau, \sigma)} \Sigma_{R,L}(\sigma_{\mp}) , \quad \Sigma_{R,L}(\tau, \sigma + \pi) = e^{i\pi\gamma} \Sigma_{R,L}(\tau, \sigma) , \quad (5.7)$$

so that the change $\gamma \rightarrow \gamma + 1$ is equivalent to the change of sign in the boundary conditions (in σ) for the free fermion field $\Sigma_{R,L}$.¹³ The fact that for $2k - 1 \leq \gamma < 2k$ the GS operators $\hat{N}_{R,L}$ take half-integer eigenvalues indicates, in particular, that in these intervals the GSO projection that must be done in the RNS approach must be the ‘reversed’ one.

In general, the model with $qR = 2n + 1$ ($\gamma = 2k + 1$ for odd w) is equivalent to the free superstring compactified on a twisted 3-torus (in the limit when the 2-torus part is replaced by 2-plane), or on a circle with antiperiodic boundary conditions for the fermions [18] (in particular, the theory with $qR = 1$ and $R < \sqrt{2\alpha'}$ will have tachyons).

The fundamental world-sheet fermions \mathcal{S} that appear in GS action (5.4) are always *periodic* in σ . This is necessary for supersymmetry of the model in the $q = 0$ limit. If one considers spinors (space-time fermions) in the space with the metric (2.6) one should assume that they satisfy the periodic boundary conditions in y for arbitrary q since this is the condition of unbroken supersymmetry in the limit $q \rightarrow 0$. This condition of correspondence with the standard superstring ‘free-theory’ limit fixes the ambiguity in the choice of a spin structure which a priori exists for all $qR = m$. Then the adequate point of view is that the breaking of supersymmetry for $qR \neq 2n$ (in particular, for $qR = 2n + 1$) is due to the non-trivial background (q -dependent spin connection) and not due to ‘special’ choice of boundary conditions.

The ‘redefined’ fermions Σ in (5.7) are effectively dependent on y and thus change phase under a shift in y -direction. For $qR = 2n + 1$ this results in antiperiodic boundary conditions for *space-time* fermions as functions of y (the space-time fields can be represented, e.g., as coefficients in expansion of a super string field $\Phi(y, \mathcal{S}, \dots)$ in powers of world-sheet fermions). As a consequence, there exists a continuous 1-parameter family of models interpolating between the standard supersymmetric $qR = 0$ model with fermions which are periodic in y and a non-supersymmetric $qR = 1$ model with fermions which are antiperiodic in y .

¹³ The Hamiltonian contains a term of the form (cf. (3.25)) $\sum_m (m - \frac{1}{2}\gamma) s_m^* s_m$, where s_m, s_m^* are standard GS fermion oscillators. The vacuum state in Fock space is defined in the standard way as being annihilated by negative frequency oscillators, i.e. $s_m |0\rangle = 0$, $m > \frac{1}{2}\gamma$. In the interval $2k - 1 \leq \gamma < 2k$ we have $\gamma - 2k + 1 < 1$, so that it is convenient to represent this term as $\sum_m [m - (\frac{1}{2}\gamma - k + \frac{1}{2}) - k + \frac{1}{2}] s_m^* s_m = \sum_r [r - (\frac{1}{2}\gamma - k + \frac{1}{2})] s_r'^* s_r'$, $s_r' \equiv s_{m-k+\frac{1}{2}}$. This will lead to the expressions for $\hat{N}_{L,R}$ and $\hat{J}_{L,R}$ in \hat{H} of the type which appear in the NS-sector in the RNS formalism, i.e. with $-\frac{1}{2}$ normal-ordering constant.

The GS formulation makes also transparent the computation of the 1-loop (torus) partition function (which will be non-vanishing for $q \neq 2nR^{-1}$ due to the absence of fermionic zero modes, i.e. the absence of supersymmetry). Indeed, the path integral computation of the GS superstring partition function is a straightforward generalization of the computation in the bosonic string model described in [11]. The first step is to expand y in eigen-values of the Laplacian on the 2-torus and redefine the fields x, x^* and $\mathcal{S}_{L,R}, \bar{\mathcal{S}}_{L,R}$ in (5.5),(5.6) to eliminate the non-zero-mode part of y from the $U(1)$ connection. The zero-mode part of y on the torus ($ds^2 = |d\sigma_1 + \tau d\sigma_2|^2$, $\tau = \tau_1 + i\tau_2$, $0 < \sigma_a \leq 1$) is $y_* = y_0 + 2\pi R(w\sigma_1 + w'\sigma_2)$, where w, w' are integer winding numbers. Integrating over the fields x, x^* and $\mathcal{S}_{L,R}^r, \bar{\mathcal{S}}_{L,R}^r$, we get a ratio of determinants of scalar operators of the type $\partial + iA$, $\bar{\partial} - i\bar{A}$ ($\partial = \frac{1}{2}(\partial_2 - \tau\partial_1)$) with constant connection

$$A = q\partial y_* = \pi\chi, \quad \bar{A} = q\bar{\partial} y_* = \pi\bar{\chi}, \quad \chi \equiv qR(w' - \tau w), \quad \bar{\chi} \equiv qR(w' - \bar{\tau} w). \quad (5.8)$$

The final expression for the partition function takes the simple form (cf. [11])¹⁴

$$Z(R, q) = cV_7 R \int \frac{d^2\tau}{\tau_2^2} \sum_{w, w'=-\infty}^{\infty} \exp(-\pi(\alpha'\tau_2)^{-1} R^2 |w' - \tau w|^2) \times \mathcal{Z}_0(\tau, \bar{\tau}; \chi, \bar{\chi}) \frac{Y^4(\tau, \bar{\tau}; \frac{1}{2}\chi, \frac{1}{2}\bar{\chi})}{Y(\tau, \bar{\tau}; \chi, \bar{\chi})}. \quad (5.9)$$

Here

$$Y(\tau, \bar{\tau}; \chi, \bar{\chi}) \equiv \frac{\det'(\partial + i\pi\chi) \det'(\bar{\partial} - i\pi\bar{\chi})}{\det'\partial \det'\bar{\partial}} = \frac{U(\tau, \bar{\tau}; \chi, \bar{\chi})}{U(\tau, \bar{\tau}; 0, 0)}, \quad (5.10)$$

$$U(\tau, \bar{\tau}; \chi, \bar{\chi}) \equiv \prod_{(n, n') \neq (0, 0)} (n' - \tau n + \chi)(n' - \bar{\tau} n + \bar{\chi}), \quad (5.11)$$

where, in the determinants, we have projected out the zero modes appearing at $\chi = \bar{\chi} = 0$ (i.e. $Y(\tau, \bar{\tau}; 0, 0) = 1$). The equivalent form of Y is (see, e.g. [28] and [11])

$$Y(\tau, \bar{\tau}; \chi, \bar{\chi}) = \exp\left[\frac{\pi(\chi - \bar{\chi})^2}{2\tau_2}\right] \frac{\theta_1(\chi|\tau)}{\chi\theta_1'(0|\tau)} \frac{\theta_1(\bar{\chi}|\bar{\tau})}{\bar{\chi}\theta_1'(0|\bar{\tau})} \quad (5.12)$$

$$= \left| \frac{\theta\left[\frac{1}{2} + qRw\right](0|\tau)}{\frac{1}{2} + qRw'} \right|^2,$$

¹⁴ Note that in the light-cone gauge the free part of the GS superstring measure is trivial (up to the τ_2^{-6} factor related to the zero modes) since the bosonic and fermionic determinants of 8 bosonic and $8_L + 8_R$ fermionic degrees of freedom cancel out (see [26,27]).

where $\theta_1(\chi|\tau) = \theta\left[\begin{smallmatrix} \frac{1}{2} \\ \frac{\tau}{2} \end{smallmatrix}\right](\chi|\tau)$.

The factor \mathcal{Z}_0 in (5.9) stands for the contributions of the integrals over the constant fields $x, x^*, \mathcal{S}_{L,R}, \bar{\mathcal{S}}_{L,R}$ (i.e. the contributions of $(n, n') = (0, 0)$ terms in the determinants) which become zero modes in the free-theory ($q = 0$) limit¹⁵

$$\mathcal{Z}_0 = \frac{(\frac{1}{2}\chi\tau_2^{-1/2})^4 (\frac{1}{2}\bar{\chi}\tau_2^{-1/2})^4}{\chi\bar{\chi}\tau_2^{-1}}, \quad (5.13)$$

i.e. $\mathcal{Z}_0 = 2^{-8}q^6R^6|w' - \tau w|^6\tau_2^{-3}$. \mathcal{Z}_0 (and thus Z) vanishes for $q \rightarrow 0$ in agreement with the restoration of supersymmetry (existence of fermionic zero modes) in this limit.¹⁶

The partition function vanishes at all points $qR = 2n$ where the fermionic determinants have zero modes (or θ_1 -functions in Y -factors in (5.9) have zeros for any w, w' , $\theta_1(0|\tau) = 0$), in agreement with the fact that the theory is trivial at these points. More generally, the theory, and, in particular, Z is periodic in q (see (3.29))

$$Z(R, q) = Z(R, q + 2nR^{-1}), \quad n = 0, \pm 1, \dots \quad (5.14)$$

For $qR = 2n + 1$ the partition function (with bosonic zero-mode singularity properly regularized) is the same as that of free superstring compactified on a circle with antiperiodic boundary conditions for space-time fermions [18] (as was already mentioned, the dependence on odd qR can be eliminated from (5.6) provided $\mathcal{S}_{R,L}, \bar{\mathcal{S}}_{R,L}$ satisfy antiperiodic boundary conditions in y).

Separating contributions of different intervals of values of w, w' in the sum in (5.9) (which correspond to different values of γ in the Hamiltonian picture after Poisson resummation) and comparing with the RNS expression it can be confirmed that the different prescriptions for the GSO projection in the different sectors discussed above are consistent with modular invariance.

Z is infrared-divergent for those values of the moduli q and R for which there are tachyonic states in the spectrum (see Section 4) and is finite for all other values of q, R (a special symmetry of a general class of tachyon-free string models with finite 1-loop cosmological constant was discussed in [29]).

¹⁵ Note that the full integrand of Z is modular invariant since the transformation of τ can be combined with a redefinition of w, w' (so that, e.g. \mathcal{Z}_0 and Y remain invariant).

¹⁶ The $q \rightarrow 0$ divergence of the bosonic ‘constant mode’ factor $\sim q^{-2}$ corresponds to the restoration of the translational invariance in the x_1, x_2 -plane in the zero magnetic field limit (this infrared divergence reproduces the factor of area of the 2-plane). This factor was projected out in [11] to get a smooth $q \rightarrow 0$ limit of Z . As is clear from the above, in the superstring theory this divergence is cancelled against the analogous fermionic ‘zero-mode’ factors which ensure the regular (zero) $q \rightarrow 0$ limit of Z .

6. $a = 1$ Melvin model and other more general static magnetic flux tube models

In the previous Sections we have discussed the simplest possible static magnetic flux tube model. More general bosonic string models were constructed in [11]. They depend on 4 real parameters (R, q, α, β) , with two ('left' and 'right') magnetic fields proportional to $q + \beta$ and $q - \alpha$ and an antisymmetric tensor proportional to $\alpha - \beta$. The most interesting subclass of these models, corresponding to the $\alpha = \beta$ case, describes *static* magnetic flux tube backgrounds. It contains the $a = \sqrt{3}$ Melvin model studied above as the special case of $\alpha = \beta = 0$ and the dilatonic $a = 1$ Melvin model as the case of $\alpha = \beta = q$. The four-dimensional geometry is given by

$$ds_4^2 = -dt^2 + d\rho^2 + F(\rho)\tilde{F}(\rho)\rho^2 d\varphi^2 + dx_3^2, \quad (6.1)$$

$$\mathcal{A}_\varphi = qF(\rho)\rho^2, \quad \mathcal{B}_\varphi = -\beta\tilde{F}(\rho)\rho^2, \quad (6.2)$$

$$e^{2(\phi-\phi_0)} = \tilde{F}(\rho), \quad e^{2\sigma} = \tilde{F}(\rho)F^{-1}(\rho), \quad F \equiv \frac{1}{1+q^2\rho^2}, \quad \tilde{F} \equiv \frac{1}{1+\beta^2\rho^2}.$$

The models with $\beta > q$ are related to the models with $\beta < q$ by the duality transformation in the Kaluza-Klein coordinate y ; more precisely, the (R, β, q) model is y -dual to $(\alpha'/R, q, \beta)$ model (so that the $a = 1$ Melvin model is the 'self-dual' point). For fixed q these models thus fill an interval $0 \leq \beta \leq q$ parametrized by β with $a = \sqrt{3}$ and $a = 1$ Melvin models being the boundary points. The non-trivial part of the corresponding Lagrangian is [11] (cf. (2.2))

$$\begin{aligned} L = & \partial_+\rho\partial_-\rho + F(\rho)\rho^2[\partial_+\varphi + (q + \beta)\partial_+y][\partial_-\varphi + (q - \beta)\partial_-y] \\ & + \partial_+y\partial_-y + \mathcal{R}(\phi_0 + \tfrac{1}{2}\ln F), \quad F^{-1} = 1 + \beta^2\rho^2. \end{aligned} \quad (6.3)$$

This model is related to the model (2.2) by the formal $O(2, 2)$ duality rotation (combination of a shift of φ by y and duality in y). Indeed, it can be formally obtained from the y -dual (3.31) to (2.2) by first changing $q \rightarrow \beta$, $\tilde{y} \rightarrow y$ in (3.31) and then shifting $\varphi \rightarrow \varphi + qy$. This explains why this bosonic model is solvable even though the ten-dimensional target space geometry is, in general, no longer flat.¹⁷ The equivalent form of (6.3) is

$$\begin{aligned} L = & \partial_+\rho\partial_-\rho + F(\rho)[\partial_+y - \beta\rho^2\partial_+\varphi'][\partial_-y + \beta\rho^2\partial_-\varphi'] \\ & + \rho^2\partial_+\varphi'\partial_-\varphi' + \mathcal{R}(\phi_0 + \tfrac{1}{2}\ln F), \end{aligned} \quad (6.4)$$

¹⁷ In [11] we used the 'rotating' coordinate system by redefining $\varphi \rightarrow \varphi - \beta t$ (the corresponding background remained static). This redefinition is not actually necessary for the solution of the model as we shall explain below.

where we have used the formal notation $\varphi' = \varphi + qy$. Introducing an auxiliary 2d vector field with components V_+, V_- we can represent (6.4) as follows, cf. (2.8) (this corresponds to ‘undoing’ the duality transformation mentioned above)

$$L = \frac{1}{2}(\partial_+ + i\beta V_+ + iq\partial_+ y)x (\partial_- - i\beta V_- - iq\partial_- y)x^* + c.c. \quad (6.5)$$

$$+ V_+ V_- - V_- \partial_+ y + V_+ \partial_- y .$$

Now it is easy to understand why the classical equations of this model are explicitly solvable in terms of free fields and the partition function is computable. In spite of the y -dependence in the first term, the equation of motion for y still imposes the constraint that V has zero field strength, $\mathcal{F}(V) = \partial_- V_+ - \partial_+ V_- = 0$: the variation over y of the first term vanishes once one uses the equation for x (this follows from the fact that qy -terms can be formally absorbed into a phase of x). Then $V_+ = C_+ + \partial_+ \tilde{y}$, $V_- = C_- + \partial_- \tilde{y}$, $C_\pm = \text{const.}$ In the equations for V_+, V_- one can again ignore the variation of the first term in (6.5) since it vanishes under $\mathcal{F}(V) = 0$. We find that $V_+ = C_+ + \partial_+ \tilde{y} = \partial_+ y$, $V_- = C_- + \partial_- \tilde{y} = -\partial_- y$. The solution of the model then effectively reduces to that of the model (2.2), the only extra non-trivial contribution being the zero mode parts of the two dual fields y and \tilde{y} . Interchanging of q and β is essentially equivalent (after solving for C_+, C_-) to interchanging y and \tilde{y} and thus momentum and winding modes.

Eliminating C_+, C_- one gets terms quartic in the angular momentum operators in the final Hamiltonian. Similar approach applies to the computation of the partition function Z . Once x, x^* have been integrated out, the integrals over the constant parts of V_+, V_- cannot be easily computed for $q\beta \neq 0$ and thus remain in the final expression [11] (see also below).

This discussion has a straightforward generalization to superstring case. A simple way to obtain the supersymmetric version of (6.3) is to start with (3.3) (with β instead of q), make the y -duality transformation,

$$L_{\text{RNS}} = \partial_+ x \partial_- x^* + \lambda_R^* \partial_+ \lambda_R + \lambda_L^* \partial_- \lambda_L \quad (6.6)$$

$$+ F(x) \left[\partial_+ y + \frac{i}{2} \beta (x \partial_+ x^* - x^* \partial_+ x + 2 \lambda_L^* \lambda_L) \right] \left[\partial_- y - \frac{i}{2} \beta (x \partial_- x^* - x^* \partial_- x + 2 \lambda_R^* \lambda_R) \right]$$

$$+ \mathcal{R}(\phi_0 + \frac{1}{2} \ln F) , \quad F^{-1} = 1 + \beta^2 x x^* ,$$

and then include the q -dependence by rotating x and λ , i.e. by replacing their derivatives by covariant derivatives with $iq\partial_\pm y$ as a connection. The action now contains the quartic fermionic terms reflecting the non-trivial (generalized) curvature of the space. The model still remains solvable. The direct analogue of (6.5) is (cf. (3.3))

$$L_{\text{RNS}} = \frac{1}{2}(\partial_+ + i\beta V_+ + iq\partial_+ y)x (\partial_- - i\beta V_- - iq\partial_- y)x^* + c.c. \quad (6.7)$$

$$\begin{aligned}
& + \lambda_R^*(\partial_+ + i\beta V_+ + iq\partial_+ y)\lambda_R + \lambda_L^*(\partial_- + i\beta V_- + iq\partial_- y)\lambda_L \\
& + V_+ V_- - V_- \partial_+ y + V_+ \partial_- y .
\end{aligned}$$

The final expressions for the Hamiltonian and partition function then look very similar to the bosonic ones (the role of fermions is just to supersymmetrize the corresponding free superstring number of states and angular momentum operators and to cancel certain normal ordering terms).

The operator quantization of the model can be performed in a similar way as in the simplest case of $a = \sqrt{3}$ Melvin model. The exact Hamiltonian corresponding to the superstring theory on the curved space-time geometry (6.1), (6.2) takes a very simple and β - q symmetric form, cf. (3.20)

$$\hat{H} = \frac{1}{2}\alpha'(-E^2 + p_\alpha^2) + \hat{N}_R + \hat{N}_L \quad (6.8)$$

$$\begin{aligned}
& + \frac{1}{2}\alpha'R^{-2}(m - qR\hat{J})^2 + \frac{1}{2}\alpha'^{-1}R^2(w - \alpha'\beta R^{-1}\hat{J})^2 - \hat{\gamma}(\hat{J}_R - \hat{J}_L) , \\
& \hat{N}_R - \hat{N}_L = mw ,
\end{aligned} \quad (6.9)$$

$$\hat{\gamma} \equiv \gamma - [\gamma] , \quad \gamma \equiv qRw + \alpha'\beta R^{-1}m - \alpha'q\beta\hat{J} , \quad (6.10)$$

where $[\gamma]$ denotes the integer part of γ (so that $0 \leq \hat{\gamma} < 1$) and the operators $\hat{N}_{R,L}$, $\hat{J}_{R,L}$ are the same as in (3.20).

The duality symmetry in the compact Kaluza-Klein direction y (which exchanges the axial and vector magnetic field parameters β and q) is now manifest. The Hamiltonian is indeed invariant under $R \leftrightarrow \alpha'R^{-1}$, $\beta \leftrightarrow q$, $m \leftrightarrow w$. The resulting expression for (mass)² is obvious from (6.8) (cf. (3.26),(3.30)). The mass formula can also be written in terms of the ‘left’ and ‘right’ magnetic field parameters and charges, $B_{L,R} \equiv q \pm \beta$, $Q_{L,R} = mR^{-1} \pm \alpha'^{-1}Rw$,

$$\alpha'M^2 = 2\hat{N}_R + 2\hat{N}_L + \frac{1}{2}\alpha'(Q_L^2 + Q_R^2) \quad (6.11)$$

$$- 2\alpha'(B_L Q_L \hat{J}_R + B_R Q_R \hat{J}_L) + \alpha'(B_L^2 \hat{J}_R + B_R^2 \hat{J}_L)\hat{J} .$$

It is clear from eq. (6.8) that all states with $\hat{J}_R - \hat{J}_L \leq \hat{N}_R + \hat{N}_L$ have positive mass squared. The only bosonic states which can be tachyonic thus lie on the first Regge trajectory with maximal value for S_R , minimal value for S_L , and zero orbital momentum, i.e. $\hat{J}_R = S_R - \frac{1}{2} = \hat{N}_R + \frac{1}{2}$, $\hat{J}_L = S_L + \frac{1}{2} = -\hat{N}_L - \frac{1}{2}$, so that $\hat{J}_R - \hat{J}_L = \hat{N}_R + \hat{N}_L + 1$. Then

$$\alpha'M^2 = 2(\hat{N}_R + \hat{N}_L)(1 - \hat{\gamma}) \quad (6.12)$$

$$+ \alpha'R^{-2}(m - qR\hat{J})^2 + \alpha'^{-1}R^2(w - \alpha'\beta R^{-1}\hat{J})^2 - 2\hat{\gamma} ,$$

which is not positive definite due to the last term $-2\hat{\gamma}$. For all other possible values of \hat{J}_R, \hat{J}_L the resulting M^2 is non-negative. In particular, all fermionic states will have $(\text{mass})^2 \geq 0$, as expected in a unitary theory. This is manifest from eq. (6.8), except for the fermions with $\hat{J}_R - \hat{J}_L = \hat{N}_R + \hat{N}_L + \frac{1}{2}$, for which there is a negative contribution $-\hat{\gamma}$ in the expression for M^2 . A close inspection of eq. (6.8) shows that $M^2 \geq 0$ is true also in this case.

From eq. (6.12) one learns that in general there are instabilities (associated with states with high spin and charge) for arbitrarily small values of the magnetic field parameters. The special case of $\beta = 0$ (or $q = 0$), corresponding to the $a = \sqrt{3}$ Melvin model discussed in Section 4, is the only exception: we have seen that in this (type II) model there are no tachyons below some *finite* value of q . Let us now consider an example which illustrates the generic pattern: the $a = 1$ Melvin model where $q = \beta$ ($B_R = 0, B_L = 2\beta$) and

$$\alpha' M^2 = 4\hat{N}_R + \alpha' Q_R^2 - 4\hat{\gamma}\hat{J}_R, \quad \gamma = \alpha'\beta Q_L - \alpha'\beta^2 \hat{J}. \quad (6.13)$$

Let us take for simplicity $R = \sqrt{\alpha'}$, and choose the states with $w = m$, $\hat{N}_L = 0$, $\hat{J}_R = \hat{N}_R + \frac{1}{2}$ and $\hat{J}_L = -\frac{1}{2}$. These states become tachyonic for β in the interval $\beta_1 < \beta < \beta_2$, with

$$\beta_{1,2} = \frac{1}{m} (1 \mp \sqrt{1 - \gamma_{\text{cr}}}) , \quad \gamma_{\text{cr}} = \frac{m^2}{m^2 + \frac{1}{2}}. \quad (6.14)$$

For large m these magnetic field parameters will be very small. Conversely, given any arbitrarily small magnetic field, there will be tachyons corresponding to states with m obeying $\beta^{-1} - 2^{-1/2} < m < \beta^{-1} + 2^{-1/2}$, where we have neglected $O(\beta)$ terms. Unlike the usual Yang-Mills type magnetic instabilities, these (being associated with higher level states) remain even after the massless level states get small masses (they can be eliminated only if the corresponding higher-spin states receive Planck-order corrections to their free-theory masses).

For generic values of the magnetic field parameters β, q the supersymmetry is broken in all these models. This can be seen directly from the spectrum. Indeed, the two magnetic fields couple to both components of the spin (S_L and S_R), which cannot be simultaneously the same for bosons and fermions. This means that bosons and fermions should get different mass shifts. When $qR = 2n_1$ and $\alpha'\beta R^{-1} = 2n_2$, $n_{1,2} = 0, \pm 1, \dots$, the theory is equivalent to the free superstring compactified on a circle (in this case $\hat{\gamma} = 0$ and, after appropriate shifts of m, w by integers, eq. (6.8) reduces to the free superstring Hamiltonian). If $qR = 2n_1 + 1$ or $\alpha'\beta R^{-1} = 2n_2 + 1$, then the necessary shift in m or w in the fermionic sector involves half-integer numbers. As discussed in the previous sections, in these cases the theory can be interpreted as a free superstring on a circle with antiperiodic boundary conditions for space-time fermions.

Finally, the partition function can be computed by a similar procedure as in the bosonic case [11]. Starting with the analogue of (6.7) in the Green-Schwarz approach we find (cf. (5.9))

$$\begin{aligned}
Z(R, q, \beta) &= cV_7 R \int \frac{d^2\tau}{\tau_2^2} \int dC d\bar{C} (\alpha' \tau_2)^{-1} \sum_{w, w'=-\infty}^{\infty} \\
&\times \exp \left(-\pi(\alpha' \beta^2 \tau_2)^{-1} [\chi \bar{\chi} - R(q + \beta)(w' - \tau w) \bar{\chi} - R(q - \beta)(w' - \bar{\tau} w) \chi \right. \\
&\quad \left. + R^2 q^2 (w' - \tau w)(w' - \bar{\tau} w)] \right) \\
&\times \mathcal{Z}_0(\tau, \bar{\tau}; \chi, \bar{\chi}) \frac{Y^4(\tau, \bar{\tau}; \frac{1}{2}\chi, \frac{1}{2}\bar{\chi})}{Y(\tau, \bar{\tau}; \chi, \bar{\chi})} , \\
\chi &\equiv 2\beta C + qR(w' - \tau w) , \quad \bar{\chi} = 2\beta \bar{C} + qR(w' - \bar{\tau} w) ,
\end{aligned} \tag{6.15}$$

where $Y(\tau, \bar{\tau}; \chi, \bar{\chi})$ and $\mathcal{Z}_0(\tau, \bar{\tau}; \chi, \bar{\chi})$ were defined in (5.12) and (5.13). The auxiliary parameters C, \bar{C} are proportional to the constant parts of V_{\pm} in (6.7). In the limit $\beta \rightarrow 0$ we recover the partition function (5.9) of the model discussed in the previous section.

The partition function (6.15) has the following symmetries (cf. (5.14)),

$$Z(R, q, \beta) = Z(\alpha' R^{-1}, \beta, q) , \tag{6.16}$$

$$Z(R, q, \beta) = Z(R, q + 2n_1 R^{-1}, \beta + 2n_2 \alpha'^{-1} R) , \quad n_{1,2} = 0, \pm 1, \dots \tag{6.17}$$

These are symmetries of the full conformal field theory (as can be seen directly from the string action in the Green-Schwarz formulation). For $qR \neq n_1$ and $\alpha' \beta R^{-1} \neq n_2$, $n_{1,2} =$ integers, there are tachyons at any value of the radius R , and the partition function contains infrared divergences. As follows from eq. (6.17), when $\alpha' \beta / R$ (or qR) is an even number, the partition function reduces to that of the $a = \sqrt{3}$ model, eq. (5.9). In particular, in the special case that both qR and $\alpha' \beta R^{-1}$ are even, the partition function is identically zero, since for these values of the magnetic field parameters the theory is equivalent to the free superstring theory. In the case when either $\alpha' \beta / R$ or qR is an odd number, the partition function is finite in a certain range of values of the radius.

7. Conclusions

The simple model considered in the main part of this paper describes type II superstring moving in a flat but topologically non-trivial 10-dimensional space. The non-trivial 3-dimensional part of this space (2.6) is a ‘twisted’ product of a 2-plane and a circle S^1 (the periodic shifts in the coordinate of S^1 being accompanied by rotations in the plane). The free continuous moduli parameters are the radius R of S^1 and the ‘twist’ q . If other

5 spatial dimensions are toroidally compactified, the model can be interpreted as corresponding to the Kaluza-Klein Melvin magnetic flux tube background in 4 dimensions (R being Kaluza-Klein radius and q being proportional to the magnetic field strength).

This model can be easily solved either in the RNS or light-cone GS approach and exhibits several interesting features. The supersymmetry is broken if $qR \neq 2n$. For $qR = 2n$ the theory is equivalent to the standard free superstring theory compactified on a circle with periodic boundary conditions for space-time fermions; for $qR = 2n + 1$ it is equivalent to the free superstring with antiperiodic boundary conditions for the fermions (the model thus continuously interpolates between these two free superstring models). The mass spectrum is invariant under $q \rightarrow q + 2nR^{-1}$ and contains tachyonic states for certain intervals of values of R and q . The one-loop vacuum amplitude $Z(R, q)$ is finite for $R > \sqrt{2\alpha'}$ but diverges for those R and q , for which there are tachyonic states in the spectrum.

The presence of tachyonic instabilities for certain finite values of R and q is not surprising in view of the magnetic interpretation of this model. This perturbative instability of the Kaluza-Klein Melvin background as a solution of superstring theory may be more serious than its potential non-perturbative instabilities discussed at field-theory level in [5].¹⁸

We have seen that the superstring versions of more general static magnetic flux tube models of [11] (which depend on compactification radius, vector and axial magnetic field parameters R, q and β) have analogous properties. In particular, supersymmetry is broken for all of these models (for generic values of β, q). These more general models reduce to the free superstring theory when both qR and $\alpha'\beta R^{-1}$ are even integers. The bosonic string partition function [11] has the following symmetries: $Z(R, q, \beta) = Z(\alpha'R^{-1}, \beta, q)$ and $Z(R, q, \beta) = Z(R, q + n_1 R^{-1}, \beta + n_2 \alpha'^{-1} R)$, $n_{1,2} = 0, \pm 1, \dots$. The same symmetries are present also in the superstring case, with the replacement $n_{1,2} \rightarrow 2n_{1,2}$ (the case of odd integers n_1, n_2 is again equivalent to the theory with antiperiodic fermions).

A common feature of all these models is the appearance of tachyonic instabilities associated with states on the first Regge trajectory. All other Regge trajectories are tachyon-free. This should be a universal feature of all static backgrounds in superstring theory. Indeed, this fact is related to unitarity (implying the absence of ‘fermionic tachyons’).¹⁹

¹⁸ For comparison, the $S^1 \times$ (Minkowski space) Kaluza-Klein vacuum is perturbatively stable but may be unstable at the non-perturbative level [30]. Let us note also that the perturbative instability of the Melvin background suggests that other, related, more general solutions, such as the Ernst geometry [4] which asymptotically reduce to Melvin, are also perturbatively unstable at the superstring-theory level.

¹⁹ Since a unitary tree-level S -matrix should correspond to a string field theory with a hermitian action, the ‘square’ of hermitian fermionic kinetic operator should be positive in any background. This translates into positivity of M^2 for the fermionic states in the case of static backgrounds.

The expression for $(\text{mass})^2$ depends on the angular momentum operator. If there were bosonic tachyons not only on the leading Regge trajectory, but also on the subleading one, then a fermionic state with the ‘intermediate’ value of the spin (but otherwise the same quantum numbers) would have $M^2 < 0$. Since this is not allowed by unitarity, in any unitary superstring model corresponding to a static background tachyonic states can only appear on the first (bosonic) Regge trajectory.

The breaking of supersymmetry in the model (3.3) is a consequence of an incompatibility between periodicity of space-time spinors in the compact Kaluza-Klein direction y and the presence of a mixing between y and the angular coordinate of 2-plane (this mixing produces a flat but globally non-trivial connection in the fermionic derivatives). Replacing the 2-plane by a *compact* space with a non-trivial isometry parametrized by a coordinate θ and mixing θ with another compact internal coordinate y , one may try to construct a similar model in which supersymmetry is broken with preservation of Lorentz symmetry in the remaining flat non-compact directions. The simplest examples of such models are string compactifications on twisted tori (or, equivalently, string analogues of the ‘Scherk-Schwarz’ [31] or ‘coordinate-dependent’ compactifications) [18,32,33,20]. Consider, e.g., the 3-torus $(x_1, x_2, y) \equiv (x_1 + 2\pi R' n_1, x_2 + 2\pi R' n_2, y + 2\pi R k)$ and twist it by imposing the condition that the shift by period in y should be accompanied by a rotation in the (x_1, x_2) -plane. For a *finite* R' the only possible rotations are by angles $\frac{1}{2}\pi n$, i.e. one may identify the points $(\theta, y) = (\theta + 2\pi n + \frac{1}{2}\pi k, y + 2\pi R k)$, $\cot \theta = x_1/x_2$. The superstring theory with this flat but non-trivial 3-space as (part of) the internal space was considered in [18] (see also [32,20]) where it was found that such a twist of the torus breaks supersymmetry and leads to the existence of tachyons for $R^2 < 2\alpha'$ and a finite (for $R^2 > 2\alpha'$) non-vanishing partition function. It is easy to see that the $R' \rightarrow \infty$ limit of the Rohm model is actually equivalent to the special case $qR = \frac{1}{4}k$ of our model. The corresponding limits of the spectra and partition functions of the two models indeed agree (the case of $k = 4$ explicitly considered in [18] is equivalent to the superstring compactified on a circle with antiperiodic boundary conditions for the fermions). Since in the present model the 2-plane is non-compact and thus the twisting angle $2\pi qR$ is arbitrary, this model continuously connects large R' limits of the models of [18] with different values of integer k .

Similar models with compact *flat* internal spaces obtained by ‘twisting’ tori always have *discrete* allowed values of the twisting parameter (a symmetry group of a lattice which generates a torus from R^N is discrete). It is of interest to study analogous ‘twistings’ of models with compact *curved* internal spaces with isometries. For example, one may consider the $SU(2) \times U(1)$ WZW model and ‘twist’ the product by shifting the two isometric Euler angles θ_L and θ_R of $SU(2)$ by the coordinate y corresponding to $U(1)$, $\theta'_L = \theta_L + q_1 y$, $\theta'_R = \theta_R + q_2 y$ (q_1, q_2 are arbitrary continuous twist parameters). The model with $q_2 = 0$ was recently discussed in [34]. The resulting action $I_{SU(2)}(\theta'_L, \theta'_R, \psi) + \int (\partial y)^2$ defines a

conformal theory (locally the 4-space is still $SU(2) \times U(1)$ group manifold). The case of $q_1 = q_2$ (or $q_1 = -q_2$) is a compact analog of the model (2.2) studied in the present paper. It is possible to show that supersymmetry is broken (in particular, there is no Killing spinors) in this ‘compact’ model for all values of the continuous parameters $q_i \neq 2nR^{-1}$ [35]. This is not, however, in contradiction with the ‘no-go’ theorem of ref. [36]. In the case of compactification on $SU(2) \times U(1)$ group space the supersymmetry is broken (in a ‘discrete’ way) already in the absence of twisting ($q_i = 0$) due to the central charge deficit (see, e.g., [37,38]). Still, analogous closed string models containing extra continuous supersymmetry-breaking ‘magnetic’ parameters may be of interest in connection with a possibility of spontaneous tree-level supersymmetry breaking in string theory.

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